Partielle Integration - Lösungen

Erarbeitung

$$f(x) = u(x) \cdot v(x) \qquad (1)$$

$$f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x) \qquad (2)$$

$$\int_a^b f'(x) = \left[f(x)\right]_a^b = \left[u(x) \cdot v(x)\right]_a^b \qquad (3)$$

$$\int_a^b f'(x) = \int_a^b u'(x) \cdot v(x) + u(x) \cdot v'(x)\, dx$$

$$= \int_a^b u'(x) \cdot v(x)\, dx + \int_a^b u(x) \cdot v'(x)\, dx \qquad (4)$$

(3)=(4):

$$\left[u(x) \cdot v(x)\right]_a^b = \int_a^b u'(x) \cdot v(x)\, dx + \int_a^b u(x) \cdot v'(x)\, dx$$

Beispiel

$$f(x) = (2x+1) \cdot \sin(x)$$

$$f'(x) = 2 \cdot \sin(x) + (2x+1) \cdot \cos(x)$$

$$\int_0^{\frac{\pi}{2}} f'(x)\, dx = \left[(2x+1) \cdot \sin(x)\right]_0^{\frac{\pi}{2}} \qquad (3)$$

$$\int_0^{\frac{\pi}{2}} f'(x)\, dx = \int_0^{\frac{\pi}{2}} 2 \cdot \sin(x)\, dx + \int_0^{\frac{\pi}{2}} (2x+1) \cdot \cos(x)\, dx \qquad (4)$$

(3)=(4):

$$\left[(2x+1) \cdot \sin(x)\right]_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} 2 \cdot \sin(x)\, dx + \int_0^{\frac{\pi}{2}} (2x+1) \cdot \cos(x)\, dx$$

$$\int_0^{\frac{\pi}{2}} (2x+1) \cdot \cos(x)\, dx = \left[(2x+1) \cdot \sin(x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \cdot \sin(x)\, dx$$

$$= \left(\left(2 \cdot \tfrac{\pi}{2} + 1\right) \cdot \sin\left(\tfrac{\pi}{2}\right) - 1 \cdot \sin(0)\right) - \left[-2\cos(x)\right]_0^{\frac{\pi}{2}}$$

$$= \pi + 1 + 2\cos\left(\tfrac{\pi}{2}\right) - 2\cos(0)$$

$$= \underline{\pi - 1}$$

Anwendung

1. $\int_0^4 2x \cdot e^x\, dx = \ldots$

<u>1. Versuch:</u> Setze $2x = u'(x)$ und $e^x = v(x)$.

$$\ldots = \left[x^2 \cdot e^x\right]_0^4 - \int_0^4 x^2 \cdot e^x\, dx$$

<u>2. Versuch:</u> Setze $2x = v(x)$ und $e^x = u'(x)$.

$$\ldots = \left[2x\cdot e^x\right]_0^4 - \int_0^4 2\cdot e^x\,dx$$
$$= 8e^4 - 0 - \left[2e^x\right]_0^4$$
$$= 8e^4 - 2e^4 + 2e^0$$
$$= \underline{6e^4 + 2}$$

Mögliche Kriterien:

- Der ganzrationale Faktor sollte als v(x) gewählt werden.
- Derjenige Faktor, der beim Ableiten "einfacher" wird, sollte als v(x) gewählt werden.
- Die Faktoren sollten so gewählt werden, dass u(x)v'(x) ein einfacherer Term als u'(x)v(x) ist.

2. $$\int_0^{\frac{\pi}{2}} \underbrace{x^2}_{u}\cdot\underbrace{\sin(x)}_{v'}\,dx = \left[\underbrace{x^2}_{u}\cdot\underbrace{(-\cos(x))}_{v}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \overbrace{\underbrace{2x}_{u'}}^{u}\cdot\overbrace{\underbrace{(-\cos(x))}_{v}}^{v'}\,dx$$
$$= \left(-\frac{\pi^2}{4}\cdot 0 + 0\cdot(-1)\right) - \left(\left[\underbrace{2x}_{u}\cdot\underbrace{(-\sin(x))}_{v}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{2}_{u'}\cdot\underbrace{(-\sin(x))}_{v}\,dx\right)$$
$$= 0 - \left(-\pi\cdot 1 + 0\cdot 0 - \left[2\cos(x)\right]_0^{\frac{\pi}{2}}\right)$$
$$= -(-\pi - 0 + 2)$$
$$= \pi - 2$$

Dies passiert, falls man die Faktoren im 2. Schritt tauscht:

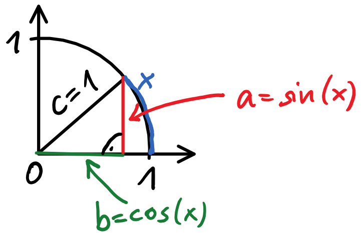
$$\int_0^{\frac{\pi}{2}} \underbrace{x^2}_{u}\cdot\underbrace{\sin(x)}_{v'}\,dx = \left[\underbrace{x^2}_{u}\cdot\underbrace{(-\cos(x))}_{v}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \overbrace{\underbrace{2x}_{u'}}^{v'}\cdot\overbrace{\underbrace{(-\cos(x))}_{v}}^{u}\,dx$$
$$= \left(-\frac{\pi}{4}^2\cdot 0 + 0\cdot(-1)\right) - \left(\left[\underbrace{x^2}_{v}\cdot\underbrace{(-\cos(x))}_{u}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{x^2}_{v}\cdot\underbrace{\sin(x)}_{u'}\,dx\right)$$
$$= 0 - 0 + \int_0^{\frac{\pi}{2}} x^2\cdot\sin(x)\,dx$$

3. $$\int_a^b \ln(x)\,dx = \int_a^b \underbrace{1}_{u'}\cdot\underbrace{\ln(x)}_{v}\,dx = \left[\underbrace{x}_{u}\cdot\underbrace{\ln(x)}_{v}\right]_a^b - \int_a^b \underbrace{x}_{u}\cdot\underbrace{\frac{1}{x}}_{v'}\,dx$$
$$= \left[x\cdot\ln(x)\right]_a^b - \int_a^b 1\,dx = \left[x\cdot\ln(x)\right]_a^b - \left[x\right]_a^b$$
$$= \left[x\cdot\ln(x) - x\right]_a^b$$

Eine Stammfunktion von f mit f(x) = ln(x) ist F(x) = x ln(x) - x .

4. $$\int_0^{\frac{\pi}{2}} \underbrace{\sin(x)}_{v}\cdot\underbrace{\cos(x)}_{u'}\,dx = \left[\underbrace{\sin(x)}_{v}\cdot\underbrace{\sin(x)}_{u}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{\cos(x)}_{v'}\cdot\underbrace{\sin(x)}_{u}\,dx$$
$$A = \left[\sin^2(x)\right]_0^{\frac{\pi}{2}} - A \qquad |+A$$
$$2A = \left[\sin^2(x)\right]_0^{\frac{\pi}{2}} \qquad |:2$$
$$A = \frac{1}{2}\left[\sin^2(x)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(1^2 - 0^2\right) = \underline{\underline{\frac{1}{2}}}$$

trigonometrischer Pythagoras



Pythagoras: $a^2+b^2=c^2$

$\sin^2(x)+\cos^2(x)=1$

Aufgaben

$$1.a)\ \int_0^{\frac{\pi}{4}} 2x\cdot\sin(x)\,dx = \left[2x\cdot(-\cos(x))\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\cdot(-\cos(x))\,dx$$
$$= -\frac{\pi}{2}\cdot\frac{1}{2}\sqrt{2}+0+\left[2\cdot\sin(x)\right]_0^{\frac{\pi}{4}}$$
$$= -\frac{\pi}{4}\sqrt{2}+\left(2\cdot\frac{1}{2}\sqrt{2}-0\right)$$
$$= \underline{-\frac{\pi}{4}\sqrt{2}+\sqrt{2}}$$

$$b)\ \int_0^3 \frac{2}{3}x\cdot e^{2x}\,dx = \left[\frac{2}{3}x\cdot\frac{1}{2}e^{2x}\right]_0^3 - \int_0^3 \frac{2}{3}\cdot\frac{1}{2}e^{2x}\,dx$$
$$= e^6-0-\left[\frac{1}{3}\cdot\frac{1}{2}e^{2x}\right]_0^3$$
$$= e^6-\frac{1}{6}e^6+\frac{1}{6}$$
$$= \underline{\frac{5}{6}e^6+\frac{1}{6}}$$

$$c)\ \int_0^4 x\cdot(x-2)^5\,dx = \left[x\cdot\frac{1}{6}(x-2)^6\right]_0^4 - \int_0^4 1\cdot\frac{1}{6}(x-2)^6\,dx$$
$$= \frac{4}{6}\cdot 2^6-0-\left[\frac{1}{6}\cdot\frac{1}{7}(x-2)^7\right]_0^4$$
$$= \frac{2^7}{3}-\frac{1}{42}\cdot 2^7+\frac{1}{42}\cdot(-2)^7$$
$$= 2^7\cdot\left(\frac{1}{3}-\frac{1}{42}-\frac{1}{42}\right)$$
$$= 2^7\cdot\frac{12}{42} = 2^7\cdot\frac{2}{7} = \underline{\underline{\frac{2^8}{7}}}$$

$$d)\ \int_0^{\pi} 4x\cdot\sin\left(x-\frac{\pi}{2}\right)dx = \left[4x\cdot\left(-\cos\left(x-\frac{\pi}{2}\right)\right)\right]_0^{\pi} - \int_0^{\pi} 4\cdot\left(-\cos\left(x-\frac{\pi}{2}\right)\right)dx$$
$$= -4\pi\cdot 0+0-\left[4\cdot\left(-\sin\left(x-\frac{\pi}{2}\right)\right)\right]_0^{\pi}$$
$$= -(-4-4) = \underline{\underline{8}}$$

e) $\int_{-\pi}^{\pi} (2x+3)\cos(x)\,dx = \left[(2x+3)\sin(x)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2\sin(x)\,dx$

$= (2\pi+3)\cdot 0 - (-2\pi+3)\cdot 0 - \left[-2\cos(x)\right]_{-\pi}^{\pi}$

$= 2\cdot(-1) - 2\cdot(-1) = \underline{\underline{0}}$

f) $\int_{1}^{2{,}5} 3x\sqrt{2x-1}\,dx = \left[3x\cdot \frac{2}{3}(2x-1)^{\frac{3}{2}}\cdot \frac{1}{2}\right]_1^{2{,}5} - \int_1^{2{,}5} 3\cdot \frac{2}{3}(2x-1)^{\frac{3}{2}}\cdot\frac{1}{2}\,dx$

$= \left(2{,}5\cdot 4^{\frac{3}{2}} - 1\cdot 1^{\frac{3}{2}}\right) - \left[\frac{2}{5}(2x-1)^{\frac{5}{2}}\cdot\frac{1}{2}\right]_1^{2{,}5}$

$= 20 - 1 - \left(\frac{1}{5}\cdot 4^{\frac{5}{2}} - \frac{1}{5}\cdot 1^{\frac{5}{2}}\right)$

$= 19 - \left(\frac{32}{5} - \frac{1}{5}\right) = \frac{95-31}{5} = \underline{\underline{\frac{64}{5}}}$

2. a) $\int_0^2 3x^2\cdot e^x\,dx = \left[3x^2\cdot e^x\right]_0^2 - \int_0^2 6xe^x\,dx$

$= 12e^2 - 0 - \left(\left[6xe^x\right]_0^2 - \int_0^2 6e^x\,dx\right)$

$= 12e^2 - (12e^2 - 0) + \left[6e^x\right]_0^2$

$= \underline{\underline{6e^2 - 6}}$

b) $\int_0^{\pi} x^2\cdot\cos(\tfrac{1}{2}x)\,dx = \left[x^2\cdot 2\sin(\tfrac{1}{2}x)\right]_0^{\pi} - \int_0^{\pi} 2x\cdot 2\sin(\tfrac{1}{2}x)\,dx$

$= 2\pi^2 - 0 - \left(\left[4x\cdot(-2\cos(\tfrac{1}{2}x))\right]_0^{\pi} - \int_0^{\pi} 4\cdot(-2\cos(\tfrac{1}{2}x))\,dx\right)$

$= 2\pi^2 - \left((-8\pi\cdot 0 + 0) - \left[-16\sin(\tfrac{1}{2}x)\right]_0^{\pi}\right)$

$= 2\pi^2 - (-(-16+0))$

$= \underline{\underline{2\pi^2 - 16}}$

c) $\int_0^{2{,}5} x^2\cdot(2x-5)^4\,dx = \left[x^2\cdot\frac{1}{5}(2x-5)^5\cdot\frac{1}{2}\right]_0^{2{,}5} - \int_0^{2{,}5} 2x\cdot\frac{1}{5}(2x-5)^5\cdot\frac{1}{2}\,dx$

$= 0 - 0 - \left(\left[\frac{1}{5}x\cdot\frac{1}{6}(2x-5)^6\cdot\frac{1}{2}\right]_0^{2{,}5} - \int_0^{2{,}5}\frac{1}{5}\cdot\frac{1}{6}(2x-5)^6\cdot\frac{1}{2}\,dx\right)$

$= -\left(0 - 0 - \left[\frac{1}{60}\cdot\frac{1}{7}\cdot(2x-5)^7\cdot\frac{1}{2}\right]_0^{2{,}5}\right.$

$= \frac{1}{120}\cdot\frac{1}{7}\cdot 0^7 - \frac{1}{120}\cdot\frac{1}{7}\cdot(-5)^7$

$= +\frac{1}{24}\cdot\frac{1}{7}\cdot 5^6 = \underline{\underline{\frac{5^6}{168}}}$

d) $\int_0^1 e^{-x}(x-1)^2\,dx = \left[-e^{-x}\cdot(x-1)^2\right]_0^1 - \int_0^1 -e^{-x}\cdot 2(x-1)\,dx$

$= 0 + 1\cdot(-1)^2 - \left(\left[+e^{-x}\cdot 2(x-1)\right]_0^1 - \int_0^1 e^{-x}\cdot 2\,dx\right)$

$= 1 - \left(0 - 1\cdot(-2) - \left[-2e^{-x}\right]_0^1\right)$

$= 1 - \left(2 - \left(-2e^{-1} + 2\right)\right)$

$= \underline{\underline{1 - 2e^{-1}}}$

e) $\int_0^2 21x^2\cdot\sqrt{1-0{,}5x}\,dx = \left[21x^2\cdot\frac{2}{3}\left(1-\frac{1}{2}x\right)^{\frac{3}{2}}\cdot(-2)\right]_0^2 - \int_0^2 42x\cdot\left(\frac{-4}{3}\right)\left(1-\frac{1}{2}x\right)^{\frac{3}{2}}dx$

$= 0 - 0 - \left(\left[42x\cdot\left(-\frac{4}{3}\right)\cdot\frac{2}{5}\left(1-\frac{1}{2}x\right)^{\frac{5}{2}}\cdot(-2)\right]_0^2\right.$

$\left. - \int_0^2 42\cdot\left(-\frac{4}{3}\right)\cdot\frac{2}{5}\left(1-\frac{1}{2}x\right)^{\frac{5}{2}}\cdot(-2)\,dx\right)$

$= -\left(0 - 0 - \left[\frac{14\cdot 16}{5}\cdot\frac{2}{7}\left(1-\frac{1}{2}x\right)^{\frac{7}{2}}\cdot(-2)\right]_0^2\right)$

$= -\frac{128}{5}\left(0 - 1^{\frac{7}{2}}\right) = \underline{\underline{\frac{128}{5}}}$

3. a) $\int_a^b 2x\cdot\ln(x)\,dx = \left[x^2\cdot\ln(x)\right]_a^b - \int_a^b x^2\cdot\frac{1}{x}\,dx$

$= \left[x^2\cdot\ln(x)\right]_a^b - \int_a^b x\,dx$

$= \left[x^2\cdot\ln(x) - \frac{1}{2}x^2\right]_a^b$

$\underline{F(x) = x^2\cdot\ln(x) - \frac{1}{2}x^2}$

b) $\int_a^b x^2\cdot\ln(x)\,dx = \left[\frac{1}{3}x^3\cdot\ln(x)\right]_a^b - \int_a^b \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$

$= \left[\frac{1}{3}x^3\ln(x)\right]_a^b - \int_a^b \frac{1}{3}x^2\,dx$

$= \left[\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3\right]_a^b$

$\underline{F(x) = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3}$

c) $\int_a^b \frac{1}{x^2}\ln(x)\,dx = \left[-\frac{1}{x}\cdot\ln(x)\right]_a^b - \int_a^b -\frac{1}{x}\cdot\frac{1}{x}\,dx$

$= \left[-\frac{1}{x}\ln(x) + \left(-\frac{1}{x}\right)\right]_a^b$

$\underline{F(x) = -\frac{1}{x}\ln(x) - \frac{1}{x}}$

4\. a) $\int_0^\pi \cos^2(x)\,dx = \int_0^\pi \underbrace{\cos(x)}_{u'}\cdot\underbrace{\cos(x)}_{v}\,dx = \left[\sin(x)\cdot\cos(x)\right]_0^\pi - \int_0^\pi \sin(x)\cdot(-\sin(x))\,dx$

$= 0 - 0 + \int_0^\pi \sin^2(x)\,dx$

$\sin^2(x) = 1-\cos^2(x)$

$= \int_0^\pi 1-\cos^2(x)\,dx$

$\underbrace{\int_0^\pi \cos^2(x)\,dx}_{=A} = [x]_0^\pi - \underbrace{\int_0^\pi \cos^2(x)\,dx}_{=A}$ $\quad |+A \quad |:2$

$A = \frac{1}{2}[x]_0^\pi = \underline{\underline{\frac{1}{2}\pi}}$

b) $\int_{-1}^1 \sin^2(\pi x)\,dx = \left[\sin(\pi x)\cdot\frac{1}{\pi}(-\cos(\pi x))\right]_{-1}^1 - \int_{-1}^1 \pi\cos(\pi x)\cdot\frac{1}{\pi}(-\cos(\pi x))\,dx$

$= 0 - 0 + \int_{-1}^1 \cos^2(\pi x)\,dx$

$\cos^2(\pi x) = 1-\sin^2(\pi x)$

$= \int_{-1}^1 1-\sin^2(\pi x)\,dx$

$\underbrace{\int_{-1}^1 \sin^2(\pi x)\,dx}_{=A} = [x]_{-1}^1 - \underbrace{\int_{-1}^1 \sin^2(\pi x)\,dx}_{=A}$ $\quad |+A \quad |:2$

$A = \frac{1}{2}[x]_{-1}^1 = \frac{1}{2}(1-(-1)) = \underline{\underline{1}}$

c) $\underbrace{\int_1^2 \frac{1}{x}\cdot\ln(x)\,dx}_{=A} = \left[\ln(x)\cdot\ln(x)\right]_1^2 - \underbrace{\int_1^2 \ln(x)\cdot\frac{1}{x}\,dx}_{=A}$ $\quad |+A \quad |:2$

$A = \frac{1}{2}\left[(\ln(x))^2\right]_1^2$

$= \frac{1}{2}\left((\ln(2))^2 - (\ln(1))^2\right) = \underline{\underline{\frac{1}{2}(\ln(2))^2}}$

$$d)\ \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{e^x}_{u'} \cdot \underbrace{\sin(x)}_{v}\,dx}_{=A} = \left[e^x \cdot \sin(x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{e^x}_{u'} \cdot \underbrace{\cos(x)}_{v}\,dx$$

$$= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - \left(\left[e^x \cdot \cos(x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cdot (-\sin(x))\,dx\right)$$

$$= e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - \left(0 - 0 + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cdot \sin(x)\,dx}_{=A}\right)$$

$$A = e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} - A \qquad | +A \quad | :2$$

$$A = \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right)$$