Integration durch Substitution

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Herleitung

$$H(x) = F(g(x))$$

$$h(x) = H'(x) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \left[F(g(x))\right]_a^b = F(g(b)) - F(g(a)) \qquad (1)$$

$$F(g(b)) - F(g(a)) = \left[F(u)\right]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\, du \qquad (2)$$

Substitution: $u = g(x)$

Beispiel:

$$\int_0^1 (2x^2+1)^3 \cdot 4x\, dx$$

$$= \int_1^3 u^3\, du$$

$g(x) = 2x^2+1$

$g'(x) = 4x$

Subst.: $u = 2x^2+1$

$f(u) = u^3$

Grenzen:

$u_1 = g(a) = 1$

$u_2 = g(b) = 3$

Anwendungen

4. a) lineare Substitution:

$$\int_{2,5}^5 \sqrt{2x-1}\, dx = \left[\frac{1}{2} \cdot \frac{2}{3}(2x-1)^{\frac{3}{2}}\right]_{2,5}^5 = \frac{1}{3} \cdot 9^{\frac{3}{2}} - \frac{1}{3} \cdot 4^{\frac{3}{2}} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

formal notiert:

$$\int_{2,5}^5 \sqrt{2x-1}\, dx = \int_4^9 \frac{1}{2}\sqrt{u}\, du$$

$$= \left[\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}}\right]_4^9$$

$$= \frac{1}{3} \cdot 9^{\frac{3}{2}} - \frac{1}{3} \cdot 4^{\frac{3}{2}} = \frac{19}{3}$$

Subst.: $u = 2x-1$

$$\frac{du}{dx} = 2 \quad \left| \cdot \frac{1}{2} dx\right.$$

$$\frac{1}{2} du = dx$$

Grenzen: $u_1 = 4$

$u_2 = 9$

b) logarithmische Substitution:

$$\int_0^{\ln(0,5)} \frac{e^{2x}}{4-e^{2x}}\,dx = \int_0^{\ln(0,5)} -\frac{1}{2}\cdot\frac{-2e^{2x}}{4-e^{2x}}\,dx$$

$$= \left[-\frac{1}{2}\ln\left(4-e^{2x}\right)\right]_0^{\ln(0,5)}$$

$$= -\frac{1}{2}\ln\left(4-e^{2\ln(\frac{1}{2})}\right)+\frac{1}{2}\ln(4-1)$$

$$= -\frac{1}{2}\ln\left(4-\frac{1}{4}\right)+\frac{1}{2}\ln 3$$

$$= -\frac{1}{2}\ln\left(\frac{15}{4}\right)+\frac{1}{2}\ln 3$$

$$= -\frac{1}{2}\ln 5-\frac{1}{2}\ln 3+\ln 2+\frac{1}{2}\ln 3$$

$$= -\frac{1}{2}\ln 5+\ln 2$$

formal notiert:

$$\int_0^{\ln(0,5)} \frac{e^{2x}}{4-e^{2x}}\,dx$$

$$= \int_3^{\frac{15}{4}} -\frac{1}{2}\cdot\frac{1}{u}\,du = \left[-\frac{1}{2}\ln u\right]_3^{\frac{15}{4}}$$

$$= -\frac{1}{2}\ln\left(\frac{15}{4}\right)+\frac{1}{2}\ln 3 = \ldots (s.o.)$$

$$= -\frac{1}{2}\ln 5+\ln 2$$

Subst.: $u = 4-e^{2x}$

$$\frac{du}{dx} = -2e^{2x} \quad \Big|\cdot\left(-\frac{1}{2}\right)dx$$

$$-\frac{1}{2}du = e^{2x}dx$$

Grenzen: $u_1 = 4-1 = 3$

$$u_2 = 4-e^{2\ln(0,5)} = 4-\frac{1}{4} = \frac{15}{4}$$

Aufgaben:

1a) $\int_0^{\frac{\pi}{2}} 3\sin(x)\cdot\cos(x)\,dx$

$$= \int_0^1 3u\,du = \left[\frac{3}{2}u^2\right]_0^1$$

$$= \frac{3}{2}-0 = \frac{3}{2}$$

Subst.: $u = \sin(x)$

$$\frac{du}{dx} = \cos(x)$$

$$du = \cos(x)dx$$

Grenzen: $u_1 = 0$

$u_2 = 1$

b) $\int_0^{14} \frac{1}{4+x^2} \cdot 2x\,dx$

$= \int_4^{200} \frac{1}{u}\,du = \left[\ln(u)\right]_4^{200}$

$= \ln(200) - \ln(4)$

$= 2\ln 10 + \ln 2 - 2\ln 2$

$= 2\ln 10 - \ln 2$

Subst.: $u = 4 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Grenzen: $u_1 = 4$

$u_2 = 200$

c) $\int_0^1 3x^2 e^{x^3+1}\,dx$

$= \int_1^2 e^u\,du = \left[e^u\right]_1^2$

$= e^2 - e$

Subst.: $u = x^3 + 1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

Grenzen: $u_1 = 1$

$u_2 = 2$

2a) $\int_{-1}^{2} x(1+x^2)^3\,dx$

$= \int_2^5 \frac{1}{2}u^3\,du = \left[\frac{1}{8}u^4\right]_2^5$

$= \frac{1}{8} \cdot 5^4 - \frac{1}{8} \cdot 2^4$

$= \frac{625-16}{8} = \frac{609}{8}$

Subst.: $u = 1 + x^2$

$\frac{du}{dx} = 2x \quad | \cdot \frac{1}{2}dx$

$\frac{1}{2}du = x\,dx$

Grenzen: $u_1 = 2$

$u_2 = 5$

b) $\int_{-1}^{1} \frac{-2x}{(4-3x^2)^2}\,dx$

$= \int_1^1 \frac{1}{3} \cdot \frac{1}{u^2}\,du = 0$

Subst.: $u = 4 - 3x^2$

$\frac{du}{dx} = -6x \quad | :3 \cdot dx$

$\frac{1}{3}du = -2x\,dx$

Grenzen: $u_1 = 1$

$u_2 = 1$

c) $\int_1^2 \frac{x+1}{\sqrt{3x^2+6x}}\,dx$

$= \int_9^{24} \frac{1}{6}\cdot\frac{1}{\sqrt{u}}\,du = \left[\frac{1}{3}\sqrt{u}\right]_9^{24}$

$= \frac{1}{3}\sqrt{24} - \frac{1}{3}\sqrt{9} = \frac{2}{3}\sqrt{6} - 1$

Subst.: $u = 3x^2+6x$

$\frac{du}{dx} = 6x+6 \quad |:6\cdot dx$

$\frac{1}{6}du = (x+1)\,dx$

Grenzen: $u_1 = 9$

$u_2 = 24$

d) $\int_1^e x^3 \ln(x^4)\,dx$

$= \int_1^{e^4} \frac{1}{4}\ln u\,du = \left[\frac{1}{4}(u\cdot\ln u - u)\right]_1^{e^4}$

$= \frac{1}{4}(e^4\cdot 4 - e^4) - \frac{1}{4}(1\cdot 0 - 1)$

$= \frac{3}{4}e^4 + \frac{1}{4}$

Subst.: $u = x^4$

$\frac{du}{dx} = 4x^3 \quad |:4\cdot dx$

$\frac{1}{4}du = x^3dx$

Grenzen: $u_1 = 1$

$u_2 = e^4$

3a) $\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx = \int_9^{25}\frac{1}{2\sqrt{u}}\,du$

$= \left[\sqrt{u}\right]_9^{25} = 5 - 3 = 2$

Subst.: $u = 9+x^2$

$\frac{du}{dx} = 2x \quad |:2\cdot dx$

$\frac{1}{2}du = x\,dx$

Grenzen: $u_1 = 9$

$u_2 = 25$

b) $\int_0^1 x\,e^{-x^2}dx = \int_0^{-1} -\frac{1}{2}e^u\,du$

$= \left[-\frac{1}{2}e^u\right]_0^{-1} = -\frac{1}{2e} + \frac{1}{2}$

Subst.: $u = -x^2$

$\frac{du}{dx} = -2x \quad |:(-2)\cdot dx$

$-\frac{1}{2}du = x\,dx$

Grenzen: $u_1 = 0$

$u_2 = -1$

c) $\int_0^{\ln 4} \frac{2e^x}{\sqrt{5-e^x}}\,dx = \int_4^1 \frac{-2}{\sqrt{u}}\,du$

$= \left[-4\sqrt{u}\right]_4^1 = -4 + 4\cdot 2 = 4$

Subst.: $u = 5-e^x$

$\frac{du}{dx} = -e^x \quad |\cdot(-2)dx$

$-2du = 2e^x dx$

Grenzen: $u_1 = 4$

$u_2 = 1$

$u_2 = 1$

d) $\int_1^{2e} \frac{(1+\ln(x))^2}{3x}\,dx$

$= \int_1^{\ln 2+2} \frac{1}{3} u^2 \,du = \left[\frac{1}{9}u^3\right]_1^{\ln 2+2}$

$= \frac{1}{9}(\ln 2+2)^3 - \frac{1}{9}$

Subst.: $u = 1+\ln x$

$\frac{du}{dx} = \frac{1}{x} \quad |:3\cdot dx$

$\frac{1}{3}du = \frac{1}{3x}dx$

Grenzen: $u_1 = 1$

$u_2 = 1+\ln(2e)$
$= 1+\ln 2+\ln e$
$= 2+\ln 2$

e) $\int_2^4 \sqrt{x^2(20-x^2)}\,dx$

$= \int_2^4 x\sqrt{20-x^2}\,dx$

$= \int_{16}^4 -\frac{1}{2}\sqrt{u}\,du = \left[-\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\right]_{16}^4$

$= -\frac{1}{3}\cdot 4^{\frac{3}{2}} + \frac{1}{3}\cdot 16^{\frac{3}{2}} = -\frac{8}{3}+\frac{64}{3} = \frac{56}{3}$

Subst.: $u = 20-x^2$

$\frac{du}{dx} = -2x \quad |:(-2)dx$

$-\frac{1}{2}du = x\,dx$

Grenzen: $u_1 = 16$

$u_2 = 4$

f) $\int_1^2 \sqrt{e^{3x}+e^{2x}}\,dx$

$= \int_1^2 e^x\sqrt{e^x+1}\,dx$

$= \int_{e+1}^{e^2+1} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{e+1}^{e^2+1}$

$= \frac{2}{3}\left((e^2+1)^{\frac{3}{2}} - (e+1)^{\frac{3}{2}}\right)$

Subst.: $u = e^x+1$

$\frac{du}{dx} = e^x \quad |\cdot dx$

$du = e^x dx$

Grenzen: $u_1 = e+1$

$u_2 = e^2+1$

4. a) partiell:

$$\underbrace{\int_e^{e^2} \underbrace{\frac{1}{x}}_{u'} \underbrace{\ln x}_{v}\,dx}_{=A} = \left[(\ln x)^2\right]_e^{e^2} - \underbrace{\int_e^{e^2} \ln x\cdot\frac{1}{x}\,dx}_{=A} \qquad |+A \quad |:2$$

$$A = \frac{1}{2}\left[(\ln e^2)^2 - (\ln e)^2\right] = \frac{1}{2}\cdot(4-1) = \frac{3}{2}$$

Substitution:

$$\int_e^{e^2} \frac{1}{x} \ln x \, dx = \int_1^2 u \, du$$

$$= \left[\frac{1}{2} u^2 \right]_1^2 = \frac{1}{2} \cdot 4 - \frac{1}{2} = \frac{3}{2}$$

Subst.: $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x} dx$$

Grenzen: $u_1 = 1$

$u_2 = 2$

b) partiell:

$$\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \underbrace{(\sin(x))^2}_{v} \cdot \underbrace{\cos(x)}_{u'} \, dx}_{=A} = \left[\frac{1}{2} (\sin(x))^2 \cdot \sin(x) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x) \cdot \cos(x) \cdot \sin(x) \, dx}_{= 2A}$$

$$3A = \left[\frac{1}{2} (\sin(x))^3 \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$A = \frac{1}{6} \left(1^3 - (-1)^3 \right) = \frac{1}{3}$$

Substitution:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (\sin(x))^2 \cdot \cos(x) \, dx$$

$$= \int_{-1}^{1} \frac{1}{2} u^2 \, du = \left[\frac{1}{6} u^3 \right]_{-1}^{1}$$

$$= \frac{1}{6} \cdot 1^3 - \frac{1}{6} (-1)^3 = \frac{1}{3}$$

Subst.: $u = \sin(x)$

$$\frac{du}{dx} = \cos(x)$$

$$du = \cos(x) \, dx$$

Grenzen: $u_1 = -1$

$u_2 = 1$