Integration durch Substitution der Integrationsvariablen

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1) a) $\int_0^1 \frac{1}{1+\sqrt{x}}\,dx$

$= \int_1^2 \frac{1}{u}\cdot 2(u-1)\,du$

$= \int_1^2 2-\frac{2}{u}\,du$

$= \left[2u - 2\ln|u|\right]_1^2$

$= 4 - 2\ln 2 - 2 + 2\ln 1$

$= 2 - 2\ln 2$

Subst.: $u = 1+\sqrt{x}$

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1)\,du$

Grenzen: $u_1 = 1$

$u_2 = 2$

b) $\int_1^2 \frac{x+1}{x^2+4x+4}\,dx$

$= \int_3^4 \frac{u-1}{u^2}\,du$

$= \int_3^4 \frac{1}{u} - \frac{1}{u^2}\,du$

$= \left[\ln|u| + \frac{1}{u}\right]_3^4$

$= \ln 4 + \frac{1}{4} - \ln 3 - \frac{1}{3}$

$= \ln 4 - \ln 3 - \frac{1}{12}$

Subst.: $u = x+2$

1. Mögl.: $u^2 = (x+2)^2$

$= x^2+4x+4$

2. Mögl.: $x = u-2$

$\frac{dx}{du} = 1$

$dx = du$

(und $x = u-2$ in den Term des Integrals einsetzen)

Grenzen: $u_1 = 3$

$u_2 = 4$

c) $\int_1^{\sqrt{e}} \frac{\ln x}{x(1-\ln x)}\,dx$

$= \int_1^{\frac{1}{2}} \frac{1-u}{e^{1-u}\cdot u}\cdot\left(-e^{1-u}\right)du$

$= \int_1^{\frac{1}{2}} -\frac{1}{u} + 1\,du$

Subst.: $u = 1-\ln x$

$x = e^{1-u}$

$\frac{dx}{du} = -e^{1-u}$

$dx = -e^{1-u}\,du$

Grenzen: $u_1 = 1$

$u_2 = \frac{1}{2}$

$$= \left[-\ln|u| + u\right]_1^{\frac{1}{2}}$$

$$= -\ln\left(\tfrac{1}{2}\right) + \tfrac{1}{2} + \ln 1 - 1$$

$$= \ln 2 - \tfrac{1}{2}$$

2a) $\int_0^2 \frac{x}{3-x}\,dx$

<u>Subst.:</u> $u = 3-x$

$x = 3-u$

$\frac{dx}{du} = -1$

$dx = -du$

<u>Grenzen:</u> $u_1 = 3$

$u_2 = 1$

$$= \int_3^1 \frac{3-u}{u} \cdot (-1)\,du$$

$$= \int_3^1 -\frac{3}{u} + 1\;du$$

$$= \left[-3\ln|u| + u\right]_3^1$$

$$= 0 + 1 + 3\ln 3 - 3 = 3\ln 3 - 2$$

b) $\int_1^{16} \frac{6}{2+\sqrt{x}}\,dx$

<u>Subst.:</u> $u = 2+\sqrt{x}$

$x = (u-2)^2$

$\frac{dx}{du} = 2(u-2)$

$dx = 2(u-2)\,du$

<u>Grenzen:</u> $u_1 = 3$

$u_2 = 6$

$$= \int_3^6 \frac{6}{u} \cdot 2(u-2)\,du$$

$$= \int_3^6 12 - \frac{24}{u}\,du$$

$$= \left[12u - 24\ln|u|\right]_3^6$$

$$= 72 - 24\ln 6 - 36 + 24\ln 3$$

$$= 36 - 24\ln 2 - 24\ln 3 + 24\ln 3$$

$$= 36 - 24\ln 2$$

c) $\displaystyle\int_4^9 \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

Subst.: $u = 1+\sqrt{x}$

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1)\,du$

Grenzen: $u_1 = 3$

$u_2 = 4$

$$= \int_3^4 \frac{1-(u-1)}{u}\cdot 2(u-1)\,du$$

$$= \int_3^4 \frac{2-u}{u}\cdot 2(u-1)\,du$$

$$= \int_3^4 \frac{4u-4-2u^2+2u}{u}\,du$$

$$= \int_3^4 -2u+6-\frac{4}{u}\,du$$

$$= \left[-u^2+6u-4\ln|u|\right]_3^4$$

$$= -16+24-4\ln 4+9-18+4\ln 3$$

$$= -1-4\ln 4+4\ln 3$$

3a) $\displaystyle\int_{\ln 4}^{\ln 5} \frac{e^{2x}}{(e^x-2)^2}\,dx$

Subst.: $u = e^x - 2$

$x = \ln(u+2)$

$\frac{dx}{du} = \frac{1}{u+2}$

$dx = \frac{1}{u+2}\,du$

Grenzen: $u_1 = 2$

$u_2 = 3$

$$= \int_2^3 \frac{(u+2)^2}{u^2}\cdot\frac{1}{u+2}\,du$$

$$= \int_2^3 \frac{u+2}{u^2}\,du = \int_2^3 \frac{1}{u}+\frac{2}{u^2}\,du$$

$$= \left[\ln|u| - \frac{2}{u}\right]_2^3$$

$$= \ln 3 - \frac{2}{3} - \ln 2 + 1$$

$$= \ln 3 - \ln 2 + \frac{1}{3}$$

b) $\int_4^{12} \frac{x}{\sqrt{2x+1}}\,dx$

Subst.: $u = 2x+1$

$x = \frac{1}{2}(u-1)$

$\frac{dx}{du} = \frac{1}{2}$

$dx = \frac{1}{2}\,du$

Grenzen: $u_1 = 9$

$u_2 = 25$

$= \int_9^{25} \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \cdot \frac{1}{2}\,du$

$= \int_9^{25} \frac{1}{4}\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du$

$= \left[\frac{1}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} - \frac{1}{4} \cdot 2 u^{\frac{1}{2}} \right]_9^{25}$

$= \left[\frac{1}{6}(\sqrt{u})^3 - \frac{1}{2}\sqrt{u} \right]_9^{25} = \frac{1}{6} \cdot 5^3 - \frac{1}{2} \cdot 5 - \frac{1}{6} \cdot 3^3 + \frac{1}{2} \cdot 3$

$= \frac{125}{6} - \frac{11}{2} = \frac{92}{6} = \frac{46}{3}$

c) $\int_0^{\sqrt{5}} \frac{x^3}{\sqrt{9-x^2}}\,dx$

Subst.: $u = 9 - x^2$

$x = \sqrt{9-u}$

$\frac{dx}{du} = \frac{-1}{2\sqrt{9-u}}$

$dx = \frac{-1}{2\sqrt{9-u}}\,du$

Grenzen: $u_1 = 9$

$u_2 = 4$

$= \int_9^4 \frac{\sqrt{9-u}^{\,3}}{\sqrt{u}} \cdot \frac{-1}{2\sqrt{9-u}}\,du$

$= \int_9^4 -\frac{9-u}{2\sqrt{u}}\,du$

$= \int_9^4 -\frac{9}{2\sqrt{u}} + \frac{\sqrt{u}}{2}\,du$

$= \left[-9\sqrt{u} + \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \right]_9^4 = -18 + \frac{8}{3} + 27 - 9$

$= \frac{8}{3}$

4a) $\displaystyle\int_0^{\frac{1}{2}\sqrt{2}} \sqrt{1-x^2}\,dx$

Subst.: $x = \sin(u)$

$\frac{dx}{du} = \cos(u)$

$dx = \cos(u)\,du$

Grenzen: $0 = \sin(u_1) \Rightarrow u_1 = 0$

$\frac{1}{2}\sqrt{2} = \sin(u_2) \Rightarrow u_2 = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1-\sin^2(u)} \cdot \cos(u)\,du$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\cos^2(u)} \cdot \cos(u)\,du$$

$$= \underbrace{\int_0^{\frac{\pi}{4}} \cos^2(u)\,du}_{=A} = \int_0^{\frac{\pi}{4}} \underbrace{\cos(u)}_{u'} \cdot \underbrace{\cos(u)}_{v}\,du$$

$$= \left[\sin(u)\cdot\cos(u)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin(u)\cdot(-\sin(u))\,du$$

$$= \left(\tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{2} - 0\right) + \int_0^{\frac{\pi}{4}} \underbrace{\sin^2(u)}_{=1-\cos^2(u)}\,du$$

$$= \frac{1}{2} + \int_0^{\frac{\pi}{4}} 1\,du - \int_0^{\frac{\pi}{4}} \cos^2(u)\,du$$

$$= \frac{1}{2} + \frac{\pi}{4} - \underbrace{\int_0^{\frac{\pi}{4}} \cos^2(u)\,du}_{=A}$$

$\Rightarrow \quad A = \frac{1}{2} + \frac{\pi}{4} - A \qquad |+A \quad |:2$

$A = \frac{1}{4} + \frac{\pi}{8}$

b) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2(u)}{\sqrt{4\cos^2(u)}} \cdot 2\cos(u)\,du$$

$$= \underbrace{\int_0^{\frac{\pi}{6}} 4\sin^2(u)\,du}_{=A}$$

$$= \int_0^{\frac{\pi}{6}} \underbrace{\sin(u)}_{u'} \cdot \underbrace{4\sin(u)}_{v}\,du$$

$$= \Big[-\cos(u)\cdot 4\sin(u)\Big]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} -\cos(u)\cdot 4\cos(u)\,du$$

$$= -\tfrac{1}{2}\sqrt{3}\cdot 4\cdot\tfrac{1}{2} + 1\cdot 0 + 4\cdot\int_0^{\frac{\pi}{6}} 1-\sin^2(u)\,du$$

$$= -\sqrt{3} + 4\cdot\frac{\pi}{6} - \underbrace{\int_0^{\frac{\pi}{6}} 4\cdot\sin^2(u)\,du}_{=A}$$

$$\Rightarrow A = -\sqrt{3} + \frac{2\pi}{3} - A \qquad |+A \quad |:2$$

$$A = -\frac{1}{2}\sqrt{3} + \frac{\pi}{3}$$

Subst.: $x = 2\sin(u)$

$\frac{dx}{du} = 2\cos(u)$

$dx = 2\cos(u)\,du$

$4-x^2 = 4(1-\sin^2(u))$

$= 4\cdot\cos^2(u)$

Grenzen: $0 = 2\sin(u_1) \Rightarrow u_1 = 0$

$1 = 2\sin(u_2)$

$\Rightarrow \frac{1}{2} = \sin(u_2) \Rightarrow u_2 = \frac{\pi}{6}$