Vertiefungskurs Mathematik 12

Lösungen: Aufgaben zu Taylorreihen

AUFGABE 1

a) $f(x) = e^{-x}$; $f'(x) = -e^{-x}$; $f''(x) = e^{-x}$ ... $f^{(k)}(x) = (-1)^k \cdot e^{-x}$

$f^{(k)}(0) = (-1)^k$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot x^k$$

Konvergenzradius: Quotientenkriterium

$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \frac{(n+1)!}{n!} = n+1$ ➔ Es gibt keinen Grenzwert

Also gilt: $r = \infty$

b) $g(x) = e^{x^2}$; $g'(x) = 2x \cdot e^{x^2}$; $g''(x) = e^{x^2} \cdot (2 + 4x^2)$

$g'''(x) = e^{x^2} \cdot (12x + 8x^3)$; $g^{(4)}(x) = e^{x^2} \cdot (12 + 48x^2 + 16x^4)$

$g^{(5)}(x) = e^{x^2} \cdot (120x + 160x^3 + 32x^5)$

$g^{(6)}(x) = e^{x^2} \cdot (120 + 720x^2 + 480x^4 + 64x^6)$

$g^{(k)}(0) = 0$ für ungerade k ➔ $a_k = 0$ für ungerade k

$g(0) = 1$; $g''(0) = 2$; $g^{(4)}(0) = 12$; $g^{(6)}(0) = 120$

$a_0 = \frac{1}{0!} = 1$; $a_2 = \frac{2}{2!} = 1$; $a_4 = \frac{12}{4!} = \frac{1}{2}$; $a_6 = \frac{120}{6!} = \frac{1}{6}$ ... $a_{2k} = \frac{1}{k!}$

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot x^{2k}$$

Konvergenzradius: Quotientenkriterium

$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \frac{(n+1)!}{n!} = n+1$ ➔ Es gibt keinen Grenzwert

Also gilt: $r = \infty$

c) $h(x) = \ln\left(1-\frac{x}{2}\right)$; $h'(x) = \frac{1}{x-2}$; $h''(x) = -\frac{1}{(x-2)^2}$; $h'''(x) = \frac{2}{(x-2)^3}$

$\ldots\ h^{(k)}(x) = (-1)^{k+1}\cdot\frac{(k-1)!}{(x-2)^k}$ ➔ $h^{(k)}(0) = (-1)^{k+1}\cdot\frac{(k-1)!}{(-2)^k} = -\frac{(k-1)!}{2^k}$

$$h(x) = \sum_{k=0}^{\infty} -\frac{1}{k\cdot 2^k}\cdot x^k$$

Konvergenzradius: Quotientenkriterium

$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n\cdot 2^n}}{\frac{1}{(n+1)\cdot 2^{n+1}}} = \frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n} = \frac{2\cdot(n+1)}{n}$ ➔ $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2\cdot(n+1)}{n} = 2$

Also gilt: $r = 2$

d) $i(x) = \frac{1}{1+\mathrm{x}}$; $i'(x) = -\frac{1}{(1+x)^2}$; $i''(x) = \frac{2}{(1+x)^3}$; $i'''(x) = -\frac{6}{(1+x)^4}$

$\ldots\ i^{(k)}(x) = (-1)^k\cdot\frac{k!}{(1+x)^{k+1}}$ ➔ $i^{(k)}(0) = (-1)^k\cdot k!$

$$i(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\cdot k!}{k!}\cdot x^k = \sum_{k=0}^{\infty}(-1)^k\cdot x^k$$

Konvergenzradius: Quotientenkriterium

$\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{1} = 1$ ➔ $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$

Also gilt: $r = 1$

AUFGABE 2 Bestimme jeweils das Taylorpolynom p_5 (d.h. vom Grad 5) um die Entwicklungsmitte $x_0 = 0$.

Berechne jeweils die prozentuale Abweichung von $p_5(1)$ von f(1).

a) $f(x) = \sqrt{1+\mathrm{x}} = (1+x)^{\frac{1}{2}}$; $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$; $f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$; $f^{(4)}(x) = -\frac{15}{16}(1+x)^{-\frac{7}{2}}$; $f^{(5)}(x) = \frac{105}{32}(1+x)^{-\frac{9}{2}}$

$a_0 = \frac{1}{0!} = 1$; $a_1 = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$; $a_2 = -\frac{\frac{1}{4}}{2!} = -\frac{1}{8}$; $a_3 = \frac{\frac{3}{8}}{3!} = \frac{1}{16}$; $a_4 = -\frac{\frac{15}{16}}{4!} = -\frac{5}{128}$

$a_5 = \frac{\frac{105}{32}}{5!} = \frac{7}{256}$ ➔ $p_5(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$

$f(1) = \sqrt{2}$; $p_5(1) = \frac{365}{256}$ ➔ $\frac{\frac{365}{256}-\sqrt{2}}{\sqrt{2}}\cdot 100\% \approx 0{,}818\%$

b) $f(x) = \tan(x)$; $f'(x) = \frac{1}{(\cos(x))^2}$; $f''(x) = \frac{2\sin(x)}{(\cos(x))^2}$

$f'''(x) = -\frac{4}{(\cos(x))^2} + \frac{6}{(\cos(x))^4}$; $f^{(4)}(x) = -\frac{8\sin(x)}{(\cos(x))^3} + \frac{24\sin(x)}{(\cos(x))^5}$

$f^{(5)}(x) = \frac{16}{(\cos(x))^2} - \frac{120}{(\cos(x))^4} + \frac{120}{(\cos(x))^6}$

$a_0 = \frac{0}{0!} = 0$; $a_1 = \frac{1}{1!} = 1$; $a_2 = \frac{0}{2!} = 0$; $a_3 = \frac{2}{3!} = \frac{1}{3}$; $a_4 = \frac{0}{4!} = 0$

$a_5 = \frac{16}{5!} = \frac{2}{15}$ ➔ $p_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

$p_5(1) = \frac{22}{15}$ ➔ $\frac{\tan(1) - \frac{22}{15}}{tan(1)} \cdot 100\% \approx 5{,}826\%$

c) $f(x) = \arcsin(x)$; $f'(x) = (1-x)^{-\frac{1}{2}}$; $f''(x) = x \cdot (1-x)^{-\frac{3}{2}}$

$f'''(x) = (1+2x^2) \cdot (1-x)^{-\frac{5}{2}}$; $f^{(4)}(x) = (9x + 6x^3) \cdot (1-x)^{-\frac{7}{2}}$

$f^{(5)}(x) = (9 + 72x^2 + 24x^4) \cdot (1-x)^{-\frac{9}{2}}$

$a_0 = \frac{0}{0!} = 0$; $a_1 = \frac{1}{1!} = 1$; $a_2 = \frac{0}{2!} = 0$; $a_3 = \frac{1}{3!} = \frac{1}{6}$; $a_4 = \frac{0}{4!} = 0$

$a_5 = \frac{9}{5!} = \frac{3}{40}$ ➔ $p_5(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5$

$p_5(1) = \frac{149}{120}$ ➔ $\frac{\arcsin(1) - \frac{149}{120}}{arcsin(1)} \cdot 100\% = \frac{\frac{\pi}{2} - \frac{149}{120}}{\frac{\pi}{2}} \cdot 100\% \approx 20{,}95\%$

d) $f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$; $f'(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$

$f''(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$; $f'''(x) = \cosh(x)$; $f^{(4)}(x) = \sinh(x)$

$f^{(5)}(x) = \cosh(x)$

$a_0 = \frac{0}{0!} = 0$; $a_1 = \frac{1}{1!} = 1$; $a_2 = \frac{0}{2!} = 0$; $a_3 = \frac{1}{3!} = \frac{1}{6}$; $a_4 = \frac{0}{4!} = 0$

$a_5 = \frac{1}{5!} = \frac{1}{120}$ ➔ $p_5(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5$

$p_5(1) = \frac{47}{40}$ ➔ $\frac{\sinh(1) - \frac{47}{40}}{sinh(1)} \cdot 100\% \approx 0017\%$

AUFGABE 3

a) $f(x) = x \cdot e^x$; $f'(x) = e^x + x \cdot e^x = (x+1) \cdot e^x$

$f''(x) = e^x + (x+1) \cdot e^x = (x+2) \cdot e^x$; $f'''(x) = (x+3) \cdot e^x$

$f^{(k)}(x) = (x+k) \cdot e^x$ ➔ $f^{(k)}(1) = (1+k) \cdot e^1 = e \cdot (1+k)$

$a_k = \frac{e \cdot (1+k)}{k!}$

$$f(x) = \sum_{k=0}^{\infty} \frac{e \cdot (1+k)}{k!} \cdot (x-1)^k$$

b) $g(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$; $g'(x) = \frac{e^x - e^{-x}}{2} = sin\text{h}\,(x)$

$g''(x) = \frac{e^x + e^{-x}}{2} = cos\text{h}\,(x)$; $g'''(x) = sinh\,(x)$; $g^{(4)}(x) = cos\text{h}\,(x)$

$a_k = \frac{1}{k!}$ für gerade k (d.h k = 2m) ; $a_k = 0$ für ungerade k

$$g(x) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \cdot x^{2m}$$