Vermischte Übungen

Donnerstag, 5. März 2020 08:47

a) $\int_0^{0,5} \frac{9x+3}{x^2-1}\,dx$ **Methode**: Partialbruchzerlegung

$$\frac{9x+3}{(x+1)(x-1)} = \frac{a}{x+1} + \frac{b}{x-1} \quad | \cdot (x+1)(x-1)$$

$$9x+3 = a(x-1) + b(x+1)$$

$$9x+3 = (a+b)x + (-a+b)$$

LGS: $\begin{cases} a+b=9 \\ -a+b=3 \end{cases} \Leftrightarrow \begin{cases} a+b=9 \\ 2b=12 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ b=6 \end{cases}$

$$\int_0^{0,5} \frac{9x+3}{x^2-1}\,dx = \int_0^{0,5} \frac{3}{x+1} + \frac{6}{x-1}\,dx = \left[3\ln|x+1| + 6\ln|x-1|\right]_0^{0,5}$$

$$= 3\ln\frac{3}{2} + 6\ln\frac{1}{2} - 3\ln 1 + 6\ln 1$$

$$= 3\ln 3 - 3\ln 2 + 6\ln 1 - 6\ln 2$$

$$= \underline{3\ln 3 - 9\ln 2}$$

Unbestimmtes Integral:

$$\int \frac{9x+3}{x^2-1}\,dx = \int \frac{3}{x+1} + \frac{6}{x-1}\,dx = 3\ln|x+1| + 6\ln|x-1|$$

b) $\int_0^{9} \frac{\sqrt{x}}{4+\sqrt{x}}\,dx$ **Methode**: Substitution der Integrationsvariable

Subst.: $u = 4+\sqrt{x} \quad |-4 \quad |(\ldots)^2$

$$x = (u-4)^2$$

$$\frac{dx}{du} = 2(u-4)$$

$$dx = 2(u-4)\,du$$

Grenzen: $u_1 = 4+\sqrt{0} = 4$

$u_2 = 4+\sqrt{9} = 7$

$$= \int_4^7 \frac{\sqrt{(u-4)^2}}{u}\, 2(u-4)\,du$$

$$= \int_4^7 \frac{2(u-4)^2}{u}\,du$$

$$= \int_4^7 \frac{2u^2-16u+32}{u}\,du$$

$$= \int_4^7 2u - 16 + \frac{32}{u}\,du$$

$$= \left[u^2 - 16u + 32\ln|u|\right]_4^7 = 49 - 112 + 32\ln 7 - 16 + 64 - 32\ln 4$$

$$= \underline{-15 + 32\ln 7 - 64\ln 2}$$

Unbestimmtes Integral:

$$\int \frac{\sqrt{x}}{4+\sqrt{x}}\,dx = \ldots = \int 2u - 16 + \frac{32}{u}\,du = u^2 - 16u + 32\ln|u|$$

$$= (4+\sqrt{x})^2 - 16(4+\sqrt{x}) + 32\ln(4+\sqrt{x})$$

$$= 16 + 8\sqrt{x} + x - 64 - 16\sqrt{x} + 32\ln(4+\sqrt{x})$$

$$= x - 8\sqrt{x} - 48 + 32\ln(4+\sqrt{x})$$

c) $\displaystyle\int_{-\pi}^{\pi} \underbrace{(x+3)}_{v} \cdot \underbrace{\sin(x)}_{u'}\,dx$ Methode: Partielle Integration

$$= \Big[\underbrace{(x+3)}_{v}\cdot\underbrace{(-\cos(x))}_{u}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \underbrace{1}_{v'}\cdot\underbrace{(-\cos(x))}_{u}\,dx$$

$$= (\pi+3)\cdot(+1) - (-\pi+3)\cdot(+1) + \big[\sin(x)\big]_{-\pi}^{\pi}$$

$$= \pi + 3 + \pi - 3 + 0 - 0$$

$$= \underline{\underline{2\pi}}$$

Unbestimmtes Integral:

$$\int (x+3)\sin(x)\,dx = (x+3)\cdot(-\cos(x)) + \sin(x) = -(x+3)\cos(x) + \sin(x)$$

d) $\displaystyle\int_0^{\ln 5} \frac{e^x - 1}{e^x + 1}\,dx$ Methode: Substitution der Integrationsvariablen und Partialbruchzerlegung

$$= \int_2^6 \frac{e^{\ln(u-1)} - 1}{u}\cdot\frac{1}{u-1}\,du$$

$$= \int_2^6 \frac{u-2}{u(u-1)}\,du$$

$$= \int_2^6 \frac{2}{u} - \frac{1}{u-1}\,du$$

$$= \big[2\ln|u| - \ln|u-1|\big]_2^6$$

$$= 2\ln 6 - \ln 5 - 2\ln 2 + \ln 1$$

$$= \underline{2\ln 3 - \ln 5}$$

Subst.: $u = e^x + 1 \quad | -1 \;|\ln(\ldots)$

$x = \ln(u-1)$

$\frac{dx}{du} = \frac{1}{u-1}$

$dx = \frac{1}{u-1}\,du$

Grenzen: $u_1 = e^0 + 1 = 2$

$u_2 = e^{\ln 5} + 1 = 6$

Partialbruchzerlegung:

$$\frac{u-2}{u(u-1)} = \frac{a}{u} + \frac{b}{u-1} \quad |\cdot u(u-1)$$

$$u - 2 = a(u-1) + bu$$

$$u - 2 = (a+b)u - a$$

LGS: $\left\{\begin{array}{l} a+b = 1 \\ -a = -2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} b = -1 \\ a = 2 \end{array}\right\}$

Unbestimmtes Integral:

$$\int \frac{e^x-1}{e^x+1}\,dx = \ldots = 2\ln|u| - \ln|u-1| = 2\ln(e^x+1) - \ln(e^x)$$
$$= \ln(e^x+1)^2 - x$$

e) $\underbrace{\int_{-1}^{1} \cos^2(\pi x)\,dx}_{=A}$ **Methode:** Partielle Integration mit trigon. Pythagoras

$$= \int_{-1}^{1} \underbrace{\cos(\pi x)}_{u} \cdot \underbrace{\cos(\pi x)}_{v'}\,dx = \Big[\underbrace{\cos(\pi x)}_{u} \cdot \underbrace{\tfrac{1}{\pi}\sin(\pi x)}_{v}\Big]_{-1}^{1} - \int_{-1}^{1} \underbrace{-\pi\sin(\pi x)}_{u'} \cdot \underbrace{\tfrac{1}{\pi}\sin(\pi x)}_{v}\,dx$$

$$= 0 + \int_{-1}^{1} 1 - \cos^2(\pi x)\,dx$$

$$= 2 - \underbrace{\int_{-1}^{1} \cos^2(\pi x)\,dx}_{=A}$$

$$\Rightarrow \quad A = 2 - A \qquad |+A \quad |:2$$
$$A = \underline{\underline{1}}$$

Unbestimmtes Integral:

$$\int \cos^2(\pi x)\,dx = \frac{1}{\pi} \cdot \cos(\pi x) \cdot \sin(\pi x) + \int \sin^2(\pi x)\,dx$$

$$\int \cos^2(\pi x)\,dx = \frac{1}{\pi}\cos(\pi x) \cdot \sin(\pi x) + x - \int \cos^2(\pi x)\,dx \qquad |+\int \ldots dx \quad |:2$$

$$\int \cos^2(\pi x)\,dx = \frac{1}{2\pi}\cos(\pi x) \cdot \sin(\pi x) + \frac{1}{2}x$$

f) $\int_{0{,}75}^{1} \frac{5}{(4x-5)^4}\,dx$ **Methode:** Lineare Substitution

$$= \int_{0{,}75}^{1} 5 \cdot (4x-5)^{-4}\,dx = \left[\frac{5}{4} \cdot \left(\frac{1}{-3}\right)(4x-5)^{-3}\right]_{0{,}75}^{1}$$

$$= -\frac{5}{12} \cdot \frac{1}{(-1)^3} + \frac{5}{12} \cdot \frac{1}{(-2)^3} = \frac{5}{12} - \frac{5}{12 \cdot 8} = \underline{\underline{\frac{35}{96}}}$$

Unbestimmtes Integral:

$$\int \frac{5}{(4x-5)^4}\,dx = \frac{5}{4} \cdot \left(\frac{1}{-3}\right) \cdot (4x-5)^{-3} = \frac{5}{12(4x-5)^3}$$

g) $\int_0^1 x \cdot \sin(x^2)\,dx$ **Methode: Substitution**

$= \int_0^1 \sin(u) \cdot \frac{1}{2}\,du$

$= \left[-\frac{1}{2}\cos(u)\right]_0^1$

$= -\frac{1}{2}\cos(1) + \frac{1}{2}\cdot 1$

$= \underline{\frac{1}{2} - \frac{1}{2}\cos(1)}$

Subst.: $u = x^2$

$\frac{du}{dx} = 2x \quad |\cdot dx\ :2$

$\frac{1}{2}du = x\,dx$

Grenzen: $u_1 = 0$

$u_2 = 1$

Unbestimmtes Integral:

$\int x \cdot \sin(x^2)\,dx = -\frac{1}{2}\cos(u) = -\frac{1}{2}\cos(x^2)$

h) $\int_2^3 \frac{x^2}{x-1}\,dx$ **Methode: Polynomdivision**

$= \int_2^3 x + 1 + \frac{1}{x-1}\,dx$

$= \left[\frac{1}{2}x^2 + x + \ln|x-1|\right]_2^3$

$(x^2 + 0x + 0) : (x-1) = x + 1 + \frac{1}{x-1}$

$-(x^2 - x)$

$x + 0$

$-(x - 1)$

1

$= \frac{9}{2} + 3 + \ln 2 - 2 - 2 - \ln 1 = \underline{3{,}5 + \ln 2}$

Unbestimmtes Integral:

$\int \frac{x^2}{x-1}\,dx = \int x + 1 + \frac{1}{x-1}\,dx = \frac{1}{2}x^2 + x + \ln|x-1|$